Concurrent Lines in the Napoleon-style Configurations

Waldemar Pompe

Abstract. Regular polygons constructed on the sides of an arbitrary triangle often lead to intriguing configurations with triples of concurrent lines. We state and prove two general theorems covering many such configurations. The theorems are illustrated by many examples.

1. INTRODUCTION. The well-known Napoleon theorem [1, Theorem 3.36] states that the centers of the equilateral triangles built (outwardly) on the sides of an arbitrary triangle are the vertices of an equilateral triangle. This configuration also possesses some other interesting properties. For example, the lines joining the vertices of the triangle with the centers of the opposite equilateral triangles are concurrent (see Figure 1). An analogous property holds if we replace equilateral triangles by squares (see Figure 2) or even by regular n-gons, for a fixed value of n. These relationships follow immediately from the following general theorem.

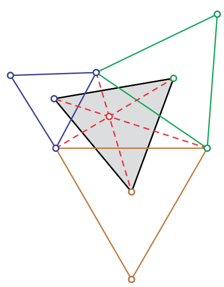

Figure 1

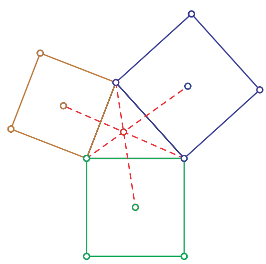

Figure 2

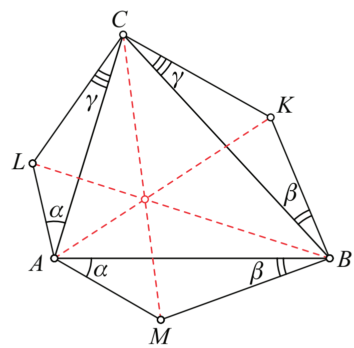

Figure 3

Theorem 1 (Isogonal Jacobi's Theorem). *Let ABC be a triangle (see Figure 3). Moreover, let BCK, CAL, and ABM be triangles constructed outside of triangle ABC. Suppose that*

$$\angle CAL = \angle BAM = \alpha, \quad \angle ABM = \angle CBK = \beta, \quad \angle BCK = \angle ACL = \gamma.$$

Then the lines AK, BL and CM are concurrent.

The most popular proof of Theorem 1 is a short calculation based on the trigonometric version of Ceva's theorem [3, page 35]. Namely, if we denote by A, B, C the measures of the respective angles of triangle ABC, then applying Ceva's theorem to triangle ABC and the three concurrent lines AK, BK, CK, we obtain

$$\frac{\sin \angle CAK}{\sin \angle KAB} \cdot \frac{\sin(B+\beta)}{\sin \beta} \cdot \frac{\sin \gamma}{\sin(C+\gamma)} = 1\,.$$

doi.org/10.1080/00029890.2019.1528824
MSC: Primary 97G40
Color versions of one or more of the figures in the article can be found online at www.tandfonline.com/uamm.

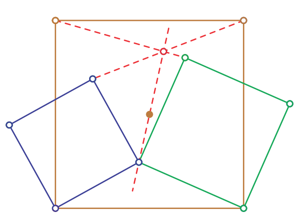

Figure 4

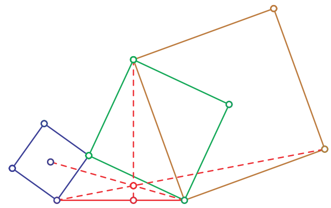

Figure 5

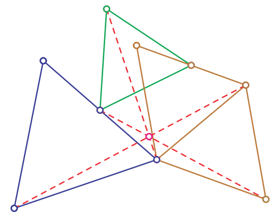

Figure 6

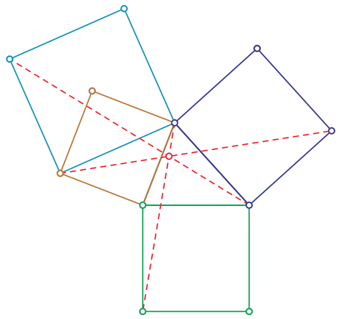

Figure 7

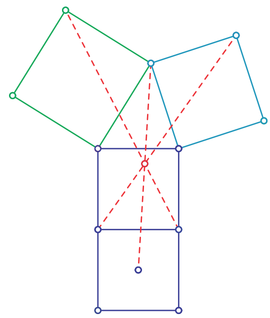

Figure 8

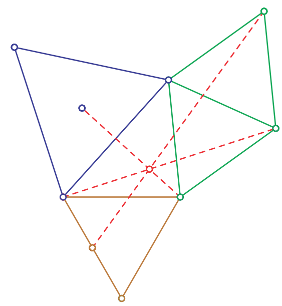

Figure 9

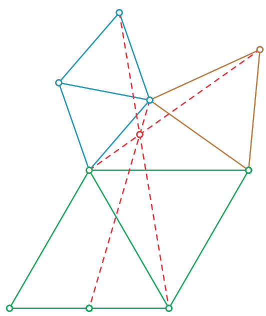

Figure 10

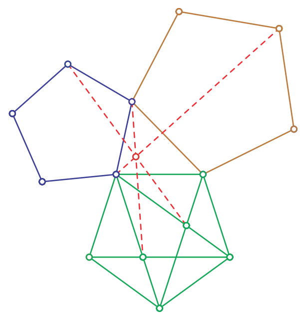

Figure 11

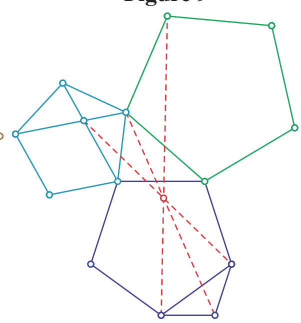

Figure 12

Writing analogous equations with respect to the vertices B, C and multiplying them together yields

$$\frac{\sin \angle CAK}{\sin \angle KAB} \cdot \frac{\sin \angle ABL}{\sin \angle LBC} \cdot \frac{\sin \angle BCM}{\sin \angle MCA} = 1\,,$$

which, again by Ceva's theorem, completes the proof of Theorem 1.

However, there are many other intriguing configurations, not covered by Theorem 1, still related to the regular polygons built on the sides of an arbitrary triangle. Some examples are presented in Figures 4–12.

The meaning of the above figures is related to the results presented in Figures 1 and 2. For example, in Figure 4 there are three squares built on sides of an *arbitrary triangle*, two of them outwardly and one inwardly. Then the dashed lines determined by the corresponding vertices of the squares and the center of the "big" square are concurrent. Similarly, in Figure 5 we again deal with an arbitrary triangle and the corresponding squares constructed, as shown. Then the dashed lines are again concurrent, where the vertical dashed line is perpendicular to the "horizontal" side of the triangle.

In this paper we present two general theorems (Theorems 2 and 3 below), from which the results presented in Figures 4–12 easily follow.

2. GENERAL CONFIGURATIONS. Let k and l be two lines. Denote by $\angle(k, l)$ the *directed angle between the lines k and l*. It is the angle about which we need to rotate the line k to get a line parallel to l. This angle is of course determined up to $180°$, i.e., two angles that differ by the integer multiple of $180°$ are considered to be the same directed angle between the lines.

It is convenient to introduce directed angles between the lines if one wants to cover various configurations in one general formulation. In such situations the following well-known theorem is often useful: *The points A, B, C, D, not lying on a common line, are concyclic if and only if*

$$\angle(AB, BC) = \angle(AD, DC).$$

This property holds no matter how the points A, B, C, D are distributed in the plane.

Theorem 2. *Let ABC be an arbitrary triangle (see Figure 13). We construct two other triangles ABD and ADM and set*

$$\alpha = \angle(DB, BA), \quad \beta = \angle(MA, AD), \quad \gamma = \angle(BD, DA), \quad \delta = \angle(AD, DM).$$

Furthermore, let the points K and L be determined by the conditions

$$\angle(LC, CA) = \alpha, \quad \angle(CA, AL) = \beta, \quad \angle(BC, CK) = \gamma, \quad \angle(KB, BC) = \delta.$$

Then the lines AK, DL and CM are concurrent.

Theorem 2 enables us to make constructions like that in Figure 7. There one starts with an *arbitrary* triangle, builds squares from two sides and an isosceles right triangle and square from the third. Then lines containing specified vertices of the squares and vertices of the triangles are concurrent.

Theorem 3. *Let ABC be an arbitrary triangle and let ω be a circle passing through the points A and B (see Figure 14). Moreover. let D and E be arbitrary points lying on ω and let M be any point. Set*

$$\alpha = \angle(DE, EA), \ \beta = \angle(ME, ED), \ \gamma = \angle(BD, DE), \ \delta = \angle(ED, DM).$$

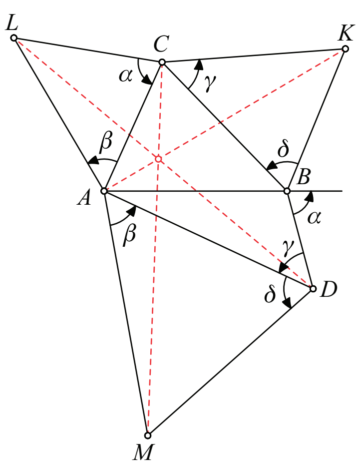

Figure 13

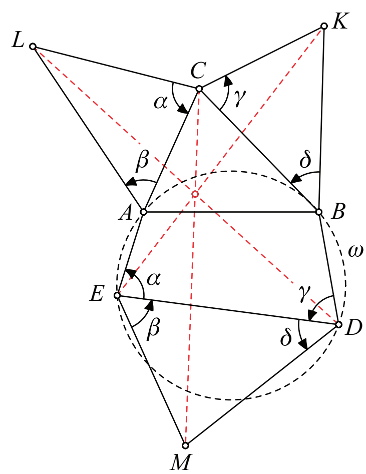

Figure 14

Let the points K and L be determined by the conditions

$$\angle(LC, CA) = \alpha,\ \angle(CA, AL) = \beta,\ \angle(BC, CK) = \gamma,\ \angle(KB, BC) = \delta.$$

Then the lines EK, DL and CM are concurrent.

Theorem 3 is the main result of this paper, from which Theorem 2 easily follows. Indeed, in the formulation of Theorem 3 the equation $\alpha = \angle(DE, EA)$ can be replaced by $\alpha = \angle(DB, BA)$. Now setting $E = A$, we may get rid of the circle ω from the formulation of Theorem 3 (as there is always a circle passing through A, B and D), giving exactly the formulation of Theorem 2.

Similarly, if we let $D = B$ in the formulation of Theorem 2, then γ becomes α. In this way we obtain Theorem 1 as a special case of Theorem 2.

Now we prove Theorems 2 and 3. Because of the remarks above, we need only prove Theorem 3. As we shall see, the proof works, if $A = E$ and/or $B = D$.

Proof of Theorem 3. Assume that the lines EK and DL intersect in S. We want to prove that points C, S, and M are collinear.

Let the circle ω intersect the line EK at the point P (see Figure 15). Since the points B, D, E, and P are concyclic, we obtain

$$\angle(BP, PK) = \angle(BD, DE) = \gamma = \angle(BC, CK).$$

This implies that the points B, K, C, and P lie on a common circle.

Furthermore, let the line DM intersect the circle ω for the second time in X. Then

$$\angle(EK, CP) = \angle(KB, BC) = \delta = \angle(ED, DX) = \angle(EK, PX).$$

Therefore the points C, P and X are collinear.

Similarly, assume that the circle ω intersects the lines DL and EM for the second time in Q and Y, respectively (see Figure 16). Then, analogous to above, we have that the points C, Q, and Y are collinear.

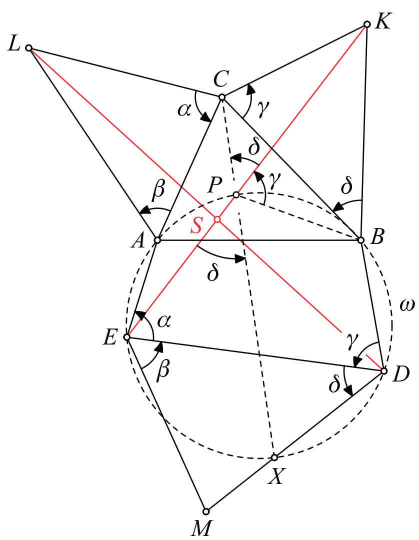

Figure 15

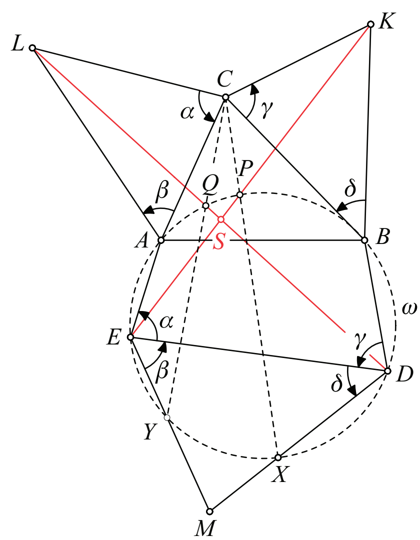

Figure 16

Finally, applying Pascal's theorem [1, Theorem 3.81] to the hexagon $XPEYQD$ inscribed in ω, we conclude that the points $XP \cap YQ = C$, $PE \cap QD = S$, and $EY \cap DX = M$ are collinear. This completes the proof of Theorem 3. ∎

Remarks.

1. If the point M lies on the circle ω, then the above proof simplifies, because it does not require the use of Pascal's theorem. In this case, the points X, Y, and M coincide; also P, Q, and S coincide. Therefore the proof is finished before invoking Pascal's theorem, as we already know that X, P, and C are collinear.

2. If $\gamma + \delta = 180°$, then the point K does not exist in the Euclidean plane. It exists however in the projective plane. Theorems 2 and 3 remain true in this situation and the above proof can be easily adapted. The only difference is that the circle containing points B, C, K, and P passes through a point at infinity (namely K), so this circle is a degenerate conic composed of the two lines BC and the line at infinity. The same remark applies if $\alpha + \beta = 180°$, i.e., if the point L lies on the line at infinity.

The following table shows which parameters should be taken in Theorems 2 and 3 in order to obtain the results presented in Figures 4–12.

Figure	**Theorem**	α	β	γ	δ
4	**3**	90°	45°	90°	45°
5	**2**	90°	90°	45°	45°
6	**2**	90°	60°	60°	60°
7	**2**	90°	45°	45°	90°
8	**3**	90°	45°	90°	45°
9	**2**	60°	30°	30°	30°
10	**2**	120°	30°	60°	60°
11	**2**	108°	36°	72°	72°
12	**3**	72°	36°	108°	36°

3. RELATED CONFIGURATIONS. As we have seen, the examples presented in Figures 4–12 are direct consequences of Theorems 2 and 3. However, there are still many other interesting configurations not covered directly by these two theorems. For example, the configurations in Figures 17 and 18 are not special cases of Theorems 2

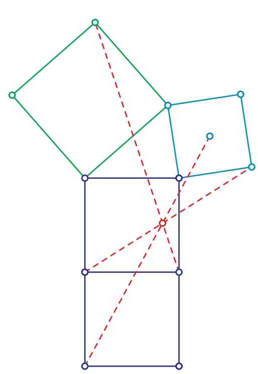

Figure 17

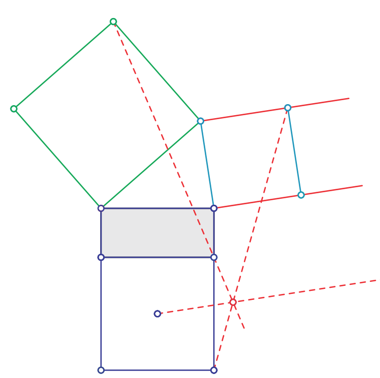

Figure 18

and 3, since none of the three concurrent dashed lines pass through a vertex of the triangle. (In Figure 18 the shaded region is an arbitrary rectangle and the three "almost horizontal" lines are parallel.) On the other hand, it turns out that Theorems 2 and 3 can still be easily applied. The following general result follows immediately from Theorem 3.

Theorem 4. *Let ABC be an arbitrary triangle. Let the points D, E, the angles α, β, γ, δ, and the points K, L, M be defined as in Theorem 3. Moreover, let the points K' and M' lie on the lines CK and EM, respectively, such that*

$$\angle(K'B, BK) = \angle(MD, DM') = \varepsilon\,.$$

Then the lines KM', $K'M$ and DL are concurrent (see Figure 19).

Proof. According to Theorem 3, the lines CM and EK meet on DL (see Figure 20). Similarly, the lines CM' and EK' also meet on DL. Therefore by Pappus's theorem [**1**, Theorem 3.51] applied to the collinear triples (C, K, K') and (E, M, M'), the lines KM' and $K'M$ meet on DL. ∎

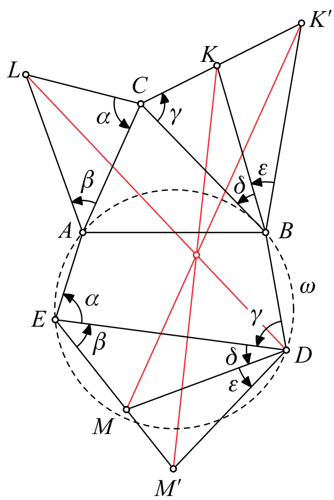

Figure 19

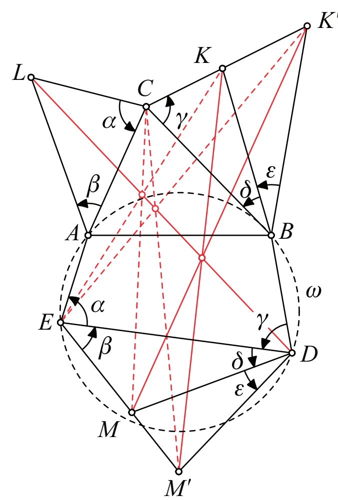

Figure 20

To obtain the configuration in Figure 17, apply Theorem 4 to

$$A = E \quad \text{and} \quad \alpha = 90^\circ,\ \beta = \gamma = \delta = \varepsilon = 45^\circ\,.$$

But to get the relationship in Figure 18, put $\alpha = \gamma = 90^\circ$ and $\beta = \delta = \varepsilon = 45^\circ$.

ACKNOWLEDGMENT. The motivation for the paper comes from the configuration presented in Figure 8, a version of which appeared in [**2**] as Problem 902. The author thanks Professor Michael de Villiers and Professor Johan Meyer for bringing the author's attention to this configuration.

REFERENCES

[1] Coxeter, H. S. M., Greitzer, S. L. (1967). *Geometry Revisited*. Washington, DC: Mathematical Association of America.
[2] Gutierrez, A. (2013). Geometry for the Land of the Incas. gogeometry.com/school-college/p902-static-triangle-squares-concurrent.htm
[3] Prasolov, V. V. (2000). *Essays on Numbers and Figures*. Mathematical World, Vol. 16, Providence, RI: American Mathematical Society.

WALDEMAR POMPE studied mathematics at the University of Warsaw. He received his Ph.D. in partial differential equations from the Technical University in Darmstadt (Germany). Since 2003 he has been working at the Institute of Mathematics at the University of Warsaw.
Institute of Mathematics, Department of Mathematics, Informatics and Mechanics, University of Warsaw, ul. Banacha 2, 02-097 Warszawa, Poland
pompe@mimuw.edu.pl

28 December 2021

In September–October 2020 I have posted several of the configurations in the Romantics of Geometry Facebook Group. Some nice solutions to those special cases have been found by Francisco Javier García Capitán, Tomasz Cieśla, Floor van Lamoen, and Nguyen Tien Dung. Here are the references:

Figure 5: Problem RG6303.
`https://www.facebook.com/groups/parmenides52/posts/3350008995112783`

Figure 6: Problem RG6284.
`https://www.facebook.com/groups/parmenides52/posts/3341149985998684`

Figure 7: Problem RG6286.
`https://www.facebook.com/groups/parmenides52/posts/3344021509044865`

Figure 9: Problem RG6272.
`https://www.facebook.com/groups/parmenides52/posts/3334592459987770`

Figure 10: Problem RG6275.
`https://www.facebook.com/groups/parmenides52/posts/3338071109639905`

Figure 11: Problem RG6418.
`https://www.facebook.com/groups/parmenides52/posts/3390979274349088`