Three reflections

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Abstract

We present several problems that can be solved in a very short way using properties of a glide reflection. In our configurations the glide reflection will be obtained as a composition of three reflections.

I dedicate this paper to the memory of Professor Edmund Puczyłowski.

1. Preliminary results

Consider a reflection about line x in a plane. For simplicity, we shall use for this reflection the same notation "x"; it should be always clear from the context, whether "x" means "line x" or "the reflection about line x". Similarly, if v is a vector, we will also denote by v the translation by vector v.

In this paper we are going to compose reflections and other isometries. As usual, by a composition "gf" of two mappings f and g we mean the mapping defined by $(gf)(X) = (g \circ f)(X) = g(f(X))$.

The following theorem is known in the theory of geometric transformations (see [1, Theorem 3.31]).

Theorem 1.1

Let a, b, c be three lines in a plane (see Fig. 1). Then there exist a unique line d and a unique vector v parallel to d, such that

$$cba = dv\,.$$

Moreover, lines a, b, c are concurrent if and only if $v = 0$.

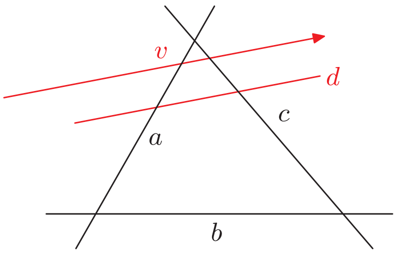

Fig. 1

Of course, general isometries f and g don't commute, i.e. $fg \neq gf$, but if vector v is parallel to line d, then it is immediately clear that they do: $dv = vd$.

The mapping $f = dv = vd$ with $d \parallel v$ is called a *glide reflection*, line d — the *mirror* or *axis* of f and v — the vector of f.

It follows from Theorem 1.1 that if lines a, b, c are concurrent, then cba reduces to a single reflection d. It turns out line d can be described in a simple geometric way. Namely, if lines a, b, and c are not parallel (see Fig. 2), then line d is determined by

$$\sphericalangle(d,c) = \sphericalangle(a,b)\,. \tag{1}$$

Here $\sphericalangle(x,y)$ denotes the oriented angle between *lines* x and y, which is defined as an angle of rotation taking line x onto a line parallel to y. It is easy to see that such an angle is defined up to 180° (the same angles differ by an integer multiple of 180°).

To see that (1) implies $d = cba$ denote by α the angles given in equality (1). Then both cd and ba are rotations with the same center $O = a \cap b$ and the same angle 2α, so they are equal mappings. From this equality we immediately obtain $d = cba$, as cc is the identity mapping.

Line d satisfying equality (1) is called an *isogonal line* to b in the angle formed by lines a and c. Note that line d can be also obtained from b by reflecting it about one of the bisectors of the angles determined by lines a and c.

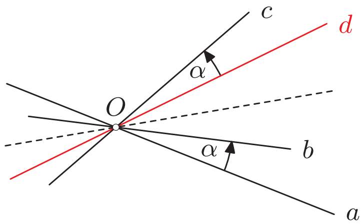

Fig. 2

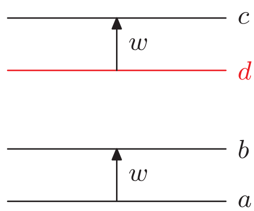

Fig. 3

Similarly, we may describe line d, if lines a, b, c are parallel (see Fig. 3). Namely, if w is a vector perpendicular to a and b that moves line a to line b, then line d should be chosen in a way that w moves line d to line c. Then both cd and ba are translations by vector $2w$, implying $cd = ba$, that is $d = cba$.

Let's get back to Theorem 1.1. There is also a geometric way to describe line d and vector v, if lines a, b, c are not concurrent and form a triangle. For this purpose we will need to introduce a *signed perimeter of an orthic triangle*.

Definition 1.2

Let ABC be a triangle. Denote by D, E, F the feet of the altitudes of triangle ABC dropped from vertices A, B, C, respectively (see Fig. 4 and Fig. 5). *A signed perimeter of the orthic triangle* of triangle ABC, denoted by $\sigma(ABC)$, is defined by:

$$\sigma(ABC) = \begin{cases} DE + EF + FD\,, & \text{if } ABC \text{ is acute-angled,} \\ -DE + EF + FD\,, & \text{if } \angle ACB \geq 90^\circ. \end{cases}$$

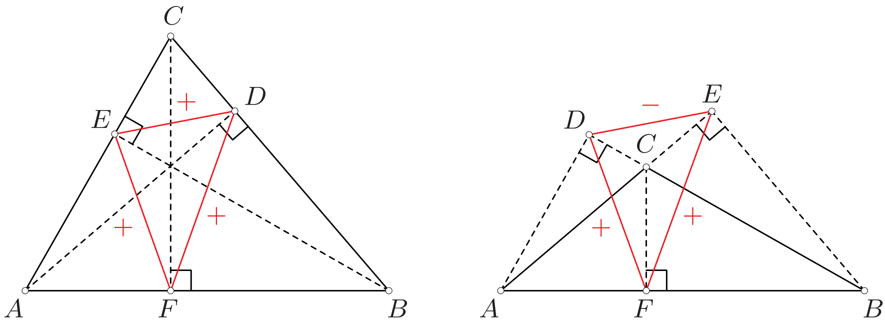

Fig. 4 Fig. 5

Theorem 1.3

Let a, b, c be lines that determine triangle ABC with $A = b \cap c$, $B = c \cap a$, $C = a \cap b$ and $\angle ABC \neq 90°$ (see Fig. 6). Denote by D, E, F the feet of the altitudes of triangle ABC taken from vertices A, B, C, respectively. If line d and vector v parallel to d are determined by the condition

$$cba = dv\,,$$

then d coincides with line DF and $|v| = \sigma(ABC)$.

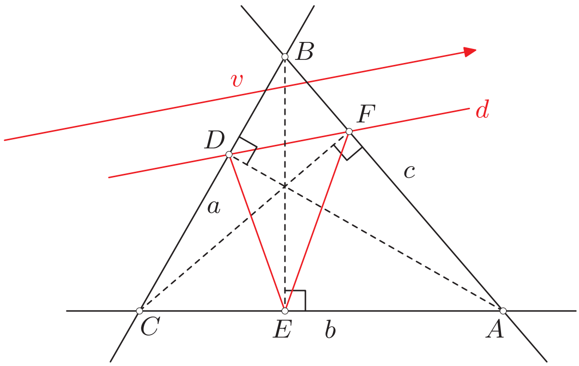

Fig. 6

Theorem 1.3 is known and can be found in the literature, e.g. [3, Section 19, Problem 13]. A similar theorem for spherical triangles can be found in [2].

Proof

The proof uses the following well-known fact from the triangle geometry: The altitudes of a triangle are the angle bisectors of the angles of the orthic triangle. More precisely, if ABC is an acute-angled triangle with AD, BE, CF as altitudes, then $\angle FEB = \angle DEB$ (see Fig. 7). The proof follows immediately from the observation that $ABDE$ and $BCEF$ are cyclic quadrilaterals. Similar formulas hold, if triangle ABC is obtuse-angled.

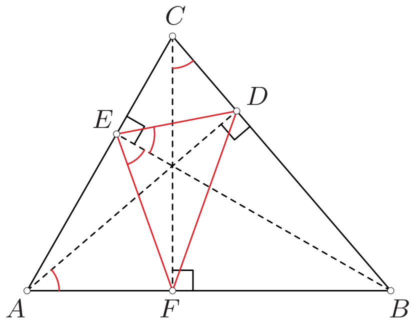

Fig. 7

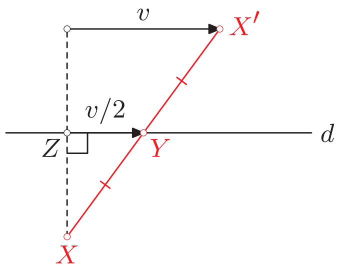

Fig. 8

We will also use the following simple observation about glide reflections. If a glide reflection $f = vd$, where d is the axis, and v is the vector of f, maps point X to X', then the midpoint Y of segment XX' lies on axis d (see Fig. 8). Moreover, if Z is the foot of the perpendicular from X onto line d, then

$$\overrightarrow{ZY} = \frac{1}{2}v\,.$$

We turn to the proof of Theorem 1.3. For simplicity, we assume that triangle ABC is acute-angled (see Fig. 9). The proof for obtuse-angled triangles is almost the same, though diagrams are a bit different (see Fig. 10, 11).

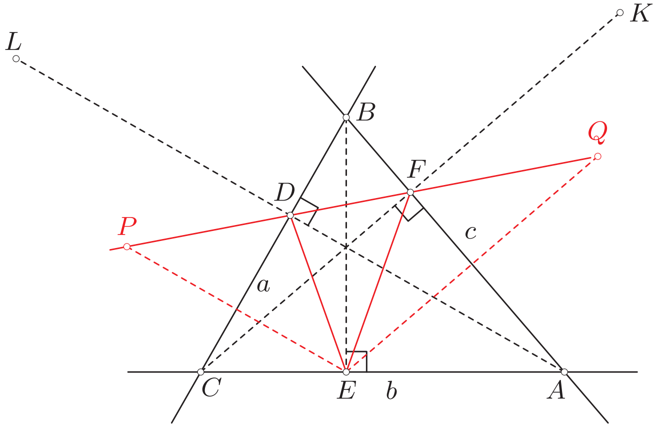

Fig. 9

Set $f = cba$. From Theorem 1.1 we know that $f = dv$, where d is a line and v is a vector parallel to d.

Let K and L be the reflections of points C and A in lines c and a, respectively. Since $f = cba$, we infer that $f(C) = K$ and $f(L) = A$. Thus D and F are the midpoints of segments $Cf(C)$ and $Lf(L)$, so D and F belong to line d. This means that line d coincides with line DF.

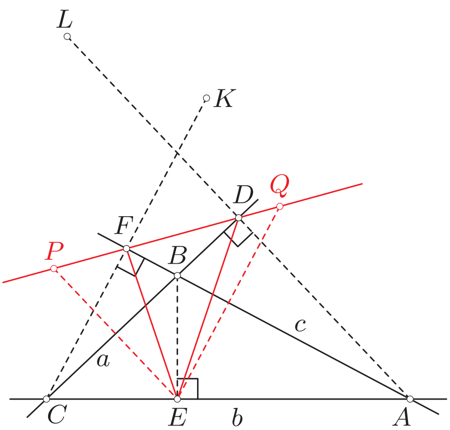

Fig. 10

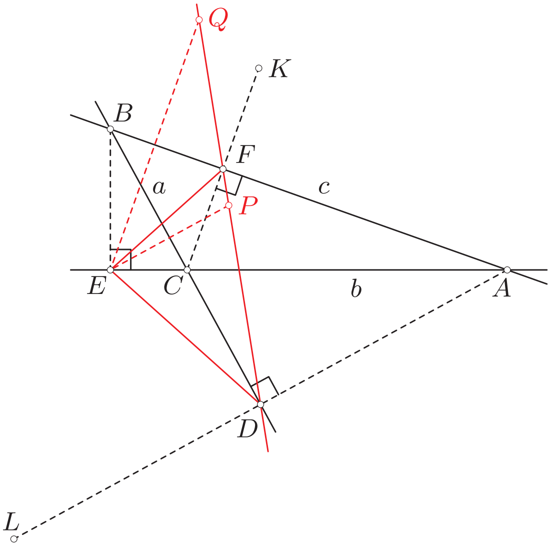

Fig. 11

Denote by P and Q the reflections of point E in lines a and c, respectively. Using the fact that we have mentioned at the beginning of the proof, we conclude that points P and Q lie on DF. Moreover f maps point P to Q, and since P lies on d, it implies $v = \overrightarrow{PQ}$. Therefore,

$$|v| = |\overrightarrow{PQ}| = \sigma(ABC),$$

which completes the proof.

Case $a \perp c$ is covered by the next theorem.

Theorem 1.4

Let a, b, c be lines that determine triangle ABC with $A = b \cap c$, $B = c \cap a$, $C = a \cap b$ and $\angle ABC = 90°$ (see Fig. 12). Denote by E the foot of the altitude of triangle ABC taken from vertex B. If line d and vector v parallel to d are determined by the condition

$$cba = dv,$$

then d coincides with the tangent line to the circumcircle of triangle ABC at point B. Moreover, $|v| = \sigma(ABC) = 2BE$.

Proof

Denote by d the tangent line do the circumcircle of triangle ABC at point B. Let M be the midpoint of segment AC and let P be the foot of the perpendicular from C onto d. Moreover, denote by x and y lines PC and BM, respectively. Then x and y are parallel, since they are perpendicular to d.

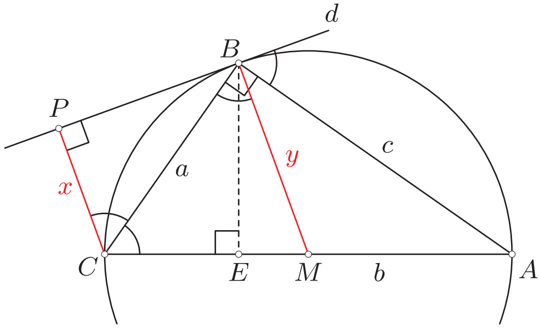

Fig. 12

Moreover, $\sphericalangle(c,d) = \sphericalangle(b,a) = \sphericalangle(a,y) = \sphericalangle(a,x)$. It follows that $ax = ba$, so $x = aba$. Therefore, $yx = (ya)ba = dcba$, so $d(yx) = cba$. Setting

$$v = 2\overrightarrow{PB},$$

we obtain $dv = cba$.

Finally, $|v| = 2PB = 2BE = \sigma(ABC)$, which completes the proof.

We conclude the preliminary results with the following observation about glide reflections.

Remark 1.5

Let d be a line and let v a vector parallel to d (see Fig. 13). Moreover, let b be a line. Then $b(dv)b$ is a glide reflection, whose axis d' and vector v' are obtained from d and v, respectively, by reflection about line b.

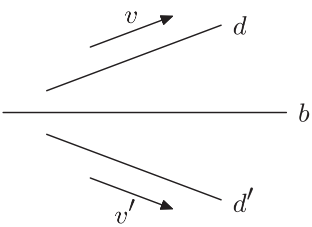

Fig. 13

For a proof simply observe that the mappings $b(dv)b$ and $d'v'$ act on a sample point X in the same way.

2. Applications

Problem 2.1

Let $ABCD$ be an arbitrary quadrilateral. The perpendicular bisectors of segments AB, BC, CD bound triangle PQR, as shown in Figure 14. Points K and L are the feet of the altitudes of triangle PQR takes from vertices Q and R, respectively. Let M be the midpoint of side AD. Prove that points K, L, and M are collinear.

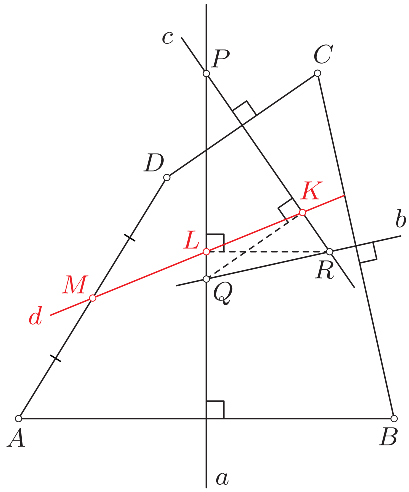

Fig. 14

Solution

Denote by a, b, c the perpendicular bisectors of segments AB, BC, CD, respectively. Then $cba = dv$, where line d coincides with KL and v is a vector parallel to KL. Observe now that cba maps A to D, so the midpoint M of AD lies on d. This completes the proof.

Problem 2.2

Let $ABCD$ be a convex quadrilateral (see Fig. 15). The bisectors x, y, z of angles $\angle A$, $\angle B$, $\angle C$, respectively, bound triangle T. Let X and Z be the feet of the altitudes of T dropped onto lines x and z, respectively. Line XZ meets side AD at point P. Prove that:

(a) $AP + BC = AB + CD + DP$;

(b) $\sigma(T) = 2DP \cdot \cos\alpha$, where $\alpha = \angle XPA$;

Solution

Reflecting line AD about lines x, y, and z, sequentially, we obtain lines AB, BC, and CD. So if we reflect point P about lines x, y, z, sequentially, we get points Q, R, S with Q on AB, R on BC, and S on CD (see Fig. 15). Moreover, $AP = AQ$, $BQ = BR$, and $CR = CS$.

Set $f = zyx$. By Theorem 1.3, we know that $f = dv$, where d coincides with line XZ and v is a vector parallel to d with $|v| = \sigma(T)$. Therefore, since P lies on d, point $S = f(P)$ must also belong to d and $\overrightarrow{PS} = v$.

Also, f maps line AD onto CD. It means line CD is obtained from AD by a reflection in line d followed by a translation. Therefore line d makes equal angles α with lines AD and CD. Thus we obtain

$$\sigma(T) = |v| = PS = 2PD\cos\alpha$$

which is (b).

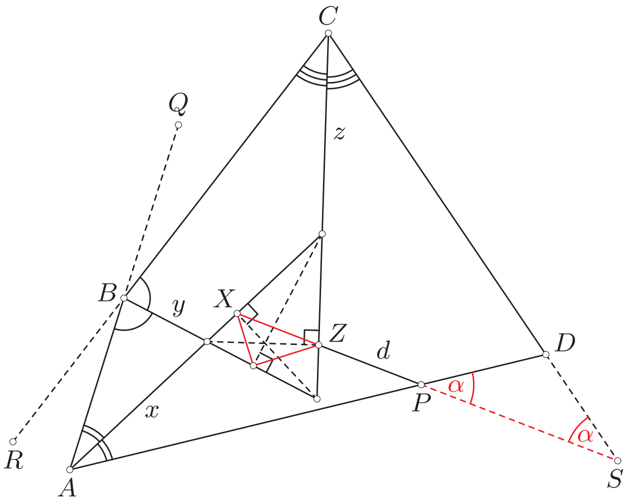

Fig. 15

To prove (a), observe that

$$AP+BC=AP+CR-BR=AQ+CS-BQ=AB+CS$$
$$=AB+CD+DP\,.$$

This completes the proof.

The next example is a very well-known theorem about the existence of the isogonal point in a triangle. The proof we are going to present is also known (see [3, Theorem 20.12] or [5]).

Problem 2.3

Let ABC be any triangle and let P be any point (see Fig. 16). Let x', y', z' be isogonal lines to lines AP, BP, CP in the angles $\angle A$, $\angle B$, $\angle C$, respectively. Prove that lines x', y', and z' are concurrent.

Solution

Denote by a, b, c lines BC, CA, AB, respectively, and by x, y, z lines AP, BP, CP, respectively.

Lines x, y, z are concurrent, so by Theorem 1.1 the mapping zyx is a reflection d. We want to prove that $z'y'x'$ is also a reflection. But

$$x'=cxb\,,\quad y'=ayc\,,\quad z'=bza\,.$$

Therefore, we obtain

$$z'y'x'=(bza)(ayc)(cxb)=b(zyx)b=bdb=d'\,,$$

where d' is a line obtained from d in reflection about line b (see Remark 1.5). Thus $z'y'x'$ is a reflection, meaning that x', y', and z' concur. This completes the proof.

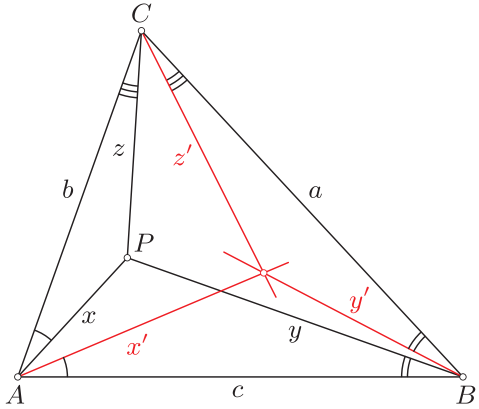

Fig. 16

From the above proof it follows that a line isogonal to y in angle $\angle(x,z)$ (which is d) and a line isogonal to y' in angle $\angle(x',z')$ (which is d') are symmetric to each other with respect to line b (see Fig. 17).

The next problem was proposed in 1952 by Victor Thébault for the American Mathematical Monthly (Problem 4470). The Proposer's published solution was based on trigonometric formulas [4].

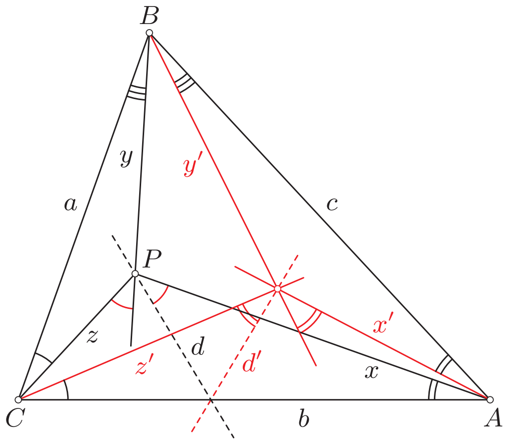

Fig. 17

Problem 2.4

Let ABC be a triangle. Assume lines x, y, z passing through vertices A, B, C, respectively, bound triangle T. Lines x', y', z' are isogonal to lines x, y, z in angles A, B, C, respectively, of the triangle ABC. Assume that lines x', y', z' bound triangle T'. Prove that the signed perimeters of the orthic triangles of T and T' are equal, i.e. $\sigma(T) = \sigma(T')$.

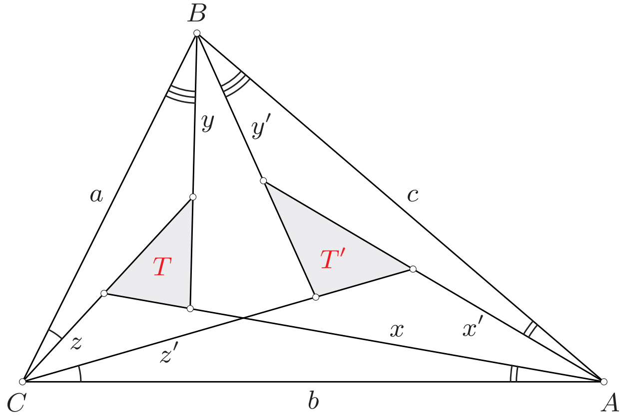

Fig. 18

Solution

Consider the mapping $f = zyx$. From Theorem 1.1 we know that mapping f can be reduced to dv, where d is a line and v is a vector parallel to d.

Denote by a, b, c lines BC, CA, AB, respectively. Since x', y', and z' are lines isogonal to x, y, and z in the corresponding angles of triangle ABC, we have:

$$x' = cxb, \quad y' = ayc, \quad z' = bza.$$

Therefore, we get

$$z'y'x' = (bza)(ayc)(cxb) = b(zyx)b = b(dv)b = d'v',$$

where d' is a line and v' a vector obtained from d and v, respectively, by reflection about line b (see Remark 1.5). Therefore, from Theorem 1.3 we immediately get

$$\sigma(T') = |v'| = |v| = \sigma(T),$$

which completes the proof.

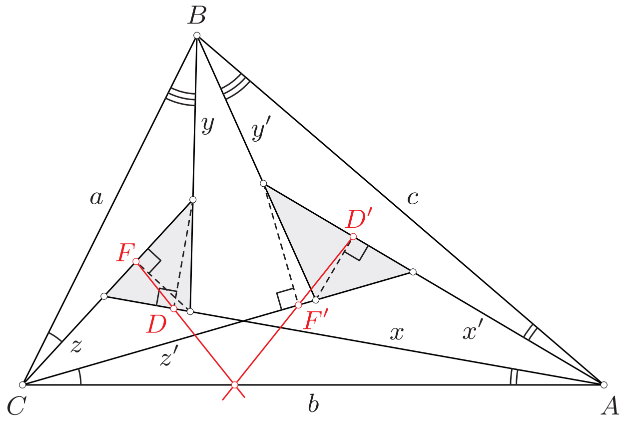

Fig. 19

Since lines d and d' are symmetric to each other with respect to line b, we have also solved the following problem.

Problem 2.5

Given triangle ABC, construct triangles T and T' as in Problem 2.4 Let D and F be the feet of the altitudes of triangle T taken onto lines x and z, respectively (see Fig. 19). Similarly, D' and F' are the feet of the altitudes of triangle T' taken onto lines x' and z', respectively. Prove that lines DF and $D'F'$ are symmetric to each other with respect to line AC. In particular, lines DF and $D'F'$ meet at point lying on line AC.

Problem 2.6.

Let $ABCDEF$ be a convex hexagon with $\angle B+\angle D+\angle F=360°$ (see Fig. 20). Denote by x, y, z, x', y', z' the perpendicular bisectors of segments AB, BC, CD, DE, EF, FA, respectively. Lines x, y, and z bound triangle T, while lines x', y', and z' bound triangle T'. Finally, denote by K, L the feet of the altitudes of T taken onto lines x, z, respectively. Similarly, K', L' are the feet of the altitudes of T' taken onto lines x', z', respectively. Prove that:

(a) points K, L, K', L' are collinear;
(b) $\sigma(T)=\sigma(T')$.

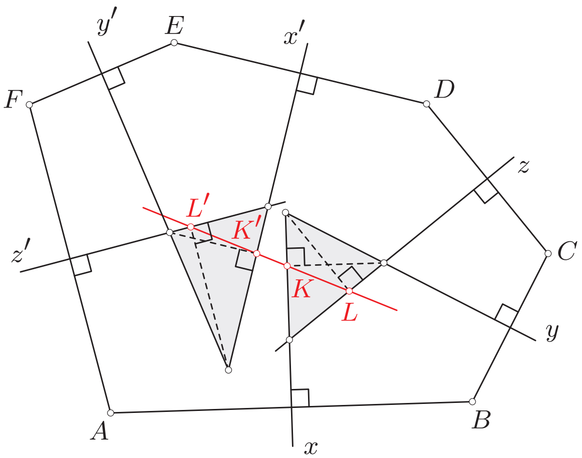

Fig. 20

Solution

Consider the mapping $f=z'y'x'zyx$. Observe that yx, $x'z$, $z'y'$ are rotations about angles

$$2\sphericalangle(x,y)=360°-2\angle B\,,\quad 2\sphericalangle(z,x')=360°-2\angle D\,,\quad 2\sphericalangle(y',z')=360°-2\angle F\,,$$

respectively. Therefore, since $\angle B+\angle D+\angle F=360°$, we infer that the above angles sum up to 360°, so mapping $f=(z'y')(x'z)(yx)$ is a translation. But since A is a fixed point of f, it follows that f is an identity.

Therefore, we obtain $z'y'x' = xyz$, Thus $z'y'x'$ and xyz are the same glide reflections, so their axes d, d' as well as vectors v, v' coincide. It follows that lines $d = KL$ and $d' = K'L'$ coincide and $\sigma(T) = |v| = |v'| = \sigma(T')$. This completes the proof.

The idea to apply the same transformation appeared earlier in Vladimir Dubrovsky's solution to the following nice problem proposed by Michael de Villiers [6]: *Prove that the intersections of the adjacent perpendicular bisectors of the sides of a hexagon with opposite sides parallel form a parallelo-hexagon, i.e. hexagon with opposite sides parallel and equal.*

References

[1] H. S. M. Coxeter, *Introduction to Geometry*, 2nd Edition, John Wiley & Sons, Inc. (1969).

[2] H. S. M. Coxeter, *The product of three reflections*, Quarterly of Applied Mathematics, Vol. 5, No. 2 (July, 1947), pp. 217–222.

[3] Clayton W. Dodge, *Euclidean Geometry and Transformations*, Dover Publications (2012).

[4] Victor Thébault, *Solution to Problem 4470*, The American Mathematical Monthly, Vol. 60, No. 4 (Apr., 1953), pp. 268-269.

[5] Vladimir Dubrovsky, *Yet another proof of the theorem on isogonal conjugation* (in Russian), Kvant 3/2016, p. 39.

[6] Vladimir Dubrovsky, *Solution to Michael de Villiers' Problem* 7741, Romantics of Geometry Facebook Group, 22 Nov 2021.
www.facebook.com/groups/parmenides52/posts/4566337916813212/

Edmund Puczyłowski (1948–2021) was a distinguished mathematician specializing in the theory non-commutative rings. He was called *Lord of the Rings* by his family and friends. In years 1998–2007 he was a chairman of the Main Committee of the Mathematical Olympiad in Poland. During this period he introduced the highest standards in the organization of the Olympiad and has initiated a very successful Junior Mathematical Olympiad in Poland. He infected everyone with passion for mathematics and mathematics education. He was a very friendly person, thinking more about the others than about himself, always ready to help.

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19 December 2021

A revised version contains a simpler proof (suggested to me by Vladimir Dubrovsky) of the second part of Theorem 1.3 and two new references [5] and [6].

Thank you, Vladimir Dubrovsky!